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RESONANCE MODES OF NEAR-CONSERVATIVE NONLINEAR SYSTEMS

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The Rauscher method is used to construct the steady-state resonance solutions of near-conservative nonautonomous multi-dimensional systems. It is assumed that the generating system has an analytic potential and admits of normal oscillations with rectilinear trajectories in configuration space. As is well known, the forced oscillations of systems with one degree of freedom in the resonance region are close to the natural oscillations of unperturbed conservative systems [1]. We present the possibility of generalizing this result to the multi-dimensional case, using the concept of normal forms of oscillations of conservative nonlinear systems [2, 3]. By selecting special types of external actions it was shown in [4] that the resonance modes possess the properties of the normal oscillations of conservative systems. For sufficiently general types of external periodic perturbations of quasi-linear systems close to Liapunov systems, Malkin [5] has exhaustively studied the periodic modes.

1. We consider the equations

$$x_s^{\cdot\cdot} = f_s(x_1, x_2, \ldots, x_n) + \varepsilon g_s(x_1, x_2, \ldots, x_n, t), \quad s = 1, 2, \ldots, n \qquad (1.1)$$

Here ε is a small parameter, f_s, g_s are analytic functions of $x_1, x_2, \ldots, x_n$; g_s is a periodic function of t of period T. We assume that the unperturbed system is conservative and admits of normal oscillations with rectilinear trajectories in configuration space: $x_{j0} = C_j x_n$ $(j = 1, 2, \ldots, n - 1)$, $C_j = \text{const}$. For a certain amplitude let the period of the natural normal oscillations also equal T. The examination is carried out in a Cartesian coordinate system in which all $C_j = 0$ and $x_{j0} = 0$. In this system we denote

$$x_n \equiv x, f_n \equiv f, g_n \equiv g$$

We look for a T-periodic solution of system (1.1) such that $x^{\cdot} = x^{\cdot}(x)$ and $t = t(x)$ are single-valued functions of x for principal values of t contained in the interval $[0, T/2]$. If such a solution exists, then the equations

$$\frac{d^2x_s}{dx^2}[x^{\cdot}(x)]^2 + \frac{dx_s}{dx}[f(x, x_1(x), \ldots, x_{n-1}(x)) + \varepsilon g(x, x_1(x), \ldots, t(x))] = \tag{1.2}$$
$$f_s(x, x_1(x), \ldots, x_{n-1}(x)) + \varepsilon g_s(x, x_1(x), \ldots, x_{n-1}(x), t(x))$$

for determining the trajectories $x_s = x_s(x)$ $(s = 1, 2, \ldots, n-1)$ have meaning.

We seek the variables $x_1, x_2, \ldots, x_{n-1}, x^{\cdot}, t$ in the form of power series in ε and x

$$x_s = \sum_{l=0}^{\infty} x_{sl}(x)\varepsilon^l, \quad x^{\cdot} = \sum_{l=0}^{\infty} x_l^{\cdot}(x)\varepsilon^l, \quad t = \sum_{l=0}^{\infty} t_l(x)\varepsilon^l \tag{1.3}$$

We choose the boundary conditions the same as for the normal oscillations of conservative systems [3]. We look for the function $x^{\cdot}(x)$ in the following form:

$$x^{\cdot 2} = 2[h - F(x, x_1(x), \ldots, x_{n-1}(x)) - \varepsilon G(x, x_1(x), \ldots, t(x))] \times \tag{1.4}$$
$$\left[1 + \sum_{j=1}^{n-1}\left(\frac{dx_j(x)}{dx}\right)^2\right]^{-1}$$

Here F is the potential of the unperturbed conservative system; the constant h and the function G are defined in the course of solving the problem.

Such values $x = X_1$ and $x = X_2$ should exist for which $x^{\cdot} = 0$. Taking (1.4) into account, we get that the relations

$$h = F(X_{1,2}, x_1(X_{1,2}), \ldots, x_{n-1}(X_{1,2})) + \varepsilon G(X_{1,2}, x_1(X_{1,2}), \ldots, t(X_{1,2})) \tag{1.5}$$

are fulfilled for the amplitude values $x = X_{1,2}$. The condition $x^{\cdot}(X_{1,2}) = 0$ together with Eqs. (1.2), leads to other relations

$$\left.\frac{dx_s}{dx}\right|_{x=X_{1,2}} \cdot [f(X_{1,2}, x_1(X_{1,2}), \ldots, x_{n-1}(X_{1,2})) + \tag{1.6}$$
$$\varepsilon g(X_{1,2}, x_1(X_{1,2}), \ldots, t(X_{1,2}))] = f_s(X_{1,2}, x_1(X_{1,2}), \ldots,$$
$$\ldots, x_{n-1}(X_{1,2})) + \varepsilon g_s(X_{1,2}, x_1(X_{1,2}), \ldots, t(X_{1,2}))$$

where $(s = 1, 2, \ldots, n-1)$. In a conservative system conditions (1.6) signify that the trajectory is orthogonal to the maximal isoenergetic surface (1.5). In a conservative system representation (1.4) follows directly from the energy integral. Since the original system (1.1) is nonconservative, conditions (1.4)–(1.6) are valid only for the steady-state resonance oscillations.

If the periodic mode trajectory has been determined, then from the equation

$$x^{\cdot\cdot} = f(x, x_1(x), \ldots, x_{n-1}(x)) + \varepsilon g(x, x_1(x), \ldots, x_{n-1}(x), t(x))$$

with initial conditions $x(0) = X_1$, $x^{\cdot}(0) = 0$, we can obtain the quadrature

$(x \leqslant X_2)$

$$t = I(x),\ I(x) = \int_{X_1}^{x} 2^{-1/2}[h - F(\xi, x_1(\xi), \ldots, x_{n-1}(\xi)) - \varepsilon G(\xi, x_1(\xi), \ldots, x_{n-1}(\xi),\quad t(\xi))]^{-1/2} d\xi \tag{1.7}$$

Hence it follows that the solution should satisfy the periodicity condition

$$T = 2I(X_2) \tag{1.8}$$

As the generating normal solution of the unperturbed conservative system of period T we choose: $x = x(t)$, $x_{j^0} = 0$ $(j = 1, 2, \ldots, n-1)$. The amplitudes X_{10}, X_{20} of the generating solution and the constant h_0 can be found from the equations

$$T = 2\int_{X_{10}}^{X_{20}} 2^{-1/2}[h - F(\xi, 0, \ldots, 0)]^{-1/2} d\xi \tag{1.9}$$

$$h_0 = F(X_{10},\,_{20},\ 0, \ldots, 0) \tag{1.10}$$

We assume that

a) the root h_0 of Eq.(1.9) and the roots X_{10}, X_{20} of Eq. (1.10) are simple;

b) the determinant $\Delta_m \neq 0$, $m = 0, 1, 2, \ldots$, (δ_s^j is the Kronecker symbol)

$$\Delta_m = \left| \delta_s^j m(m-1)\frac{2\cdot f^{(r)}(1, 0, \ldots, 0)}{r+1} + \delta_s^j m f^{(r)}(1, 0, \ldots, 0) - \frac{\partial f_s^{(r)}}{\partial x_j}(1, 0, \ldots, 0) \right| \tag{1.11}$$

where the functions $f^{(r)}$, $f_s^{(r)}$ contain only terms of the rth degree in $x, x_1, x_2, \ldots, x_{n-1}$. In the quasi-linear case $(r = 1)$ constraint (1.11) corresponds to excluding from consideration multiple frequencies in the generating system. Such a condition has been accepted when investigating Liapunov systems and near-Liapunov systems [5].

We determine the function $t_0 = t_0(x)$ by the quadrature

$$t_0 = \int_{X_{10}}^{x} 2^{-1/2}[h_0 - F(\xi, 0, \ldots, 0)]^{-1/2} d\xi, \quad x \leqslant X_{20}$$

If assumption (a) is valid, then for the principal values of t_0 the function $t_0 = t_0(x)$ is a single-valued analytic function of x [1, 6].

2. Assume that $x_{sl}(x)$, $t_l(x)$, $x_l^*(x)$, $l < k$ have been determined. In the kth approximation with respect to ε we obtain the equation

$$2\frac{d^2 x_{sk}}{dx^2}[h_0 - F(x, 0, \ldots, 0)] + \frac{dx_{sk}}{dx} f(x, 0, \ldots, 0) = \sum_{j=1}^{n-1} \frac{\partial f_s}{\partial x_j}(x, 0, \ldots, 0)\, x_{jk} + N_{sk}(x),\quad s = 1, 2, \ldots, n-1 \tag{2.1}$$

where N_{sk} are known functions depending on $x_{sl}(x)$, $t_l(x)$ $(l < k)$. Let us represent the solution of (2.1) as a power series in x

$$x_{sk} = \sum_{j=0}^{\infty} A_{sj}^{(k)} x^j \tag{2.2}$$

Substituting series (2.2) into Eqs. (2.1), we find that the coefficients $A_{sj}^{(k)}$ and $A_{sj+r}^{(k)}$ are expressed in terms of each other in a one-to-one manner. All the quantities $A_{sj}^{(k)}$ can be expressed in terms of the $2(n-1)$ coefficients $\{A_{s0}^{(k)}\}$ and $\{A_{s1}^{(k)}\}$ $(s = 1, 2, \ldots, n-1)$. The unknown coefficients are uniquely obtained as power series in $X_{1,2}$ from the boundary conditions (1.6) corresponding to the kth approximation

$$\left.\frac{dx_{sk}}{dx}\right|_{x=X_{1,2}} \cdot f(X_{1,2}, 0, \ldots, 0) = \sum_{j=1}^{n-1} \frac{\partial f_s}{\partial x_j}(X_{1,2}, 0, \ldots, 0)\, x_{jk}(X_{1,2}) + N_{sk}(X_{1,2}) \tag{2.3}$$

In this approximation the function G is determined from the equality

$$\frac{dG}{dx} = g\left(x, \sum_{l=0}^{k-1} x_{1l}\varepsilon^l, \ldots, \sum_{l=0}^{k-1} x_{n-1l}\varepsilon^l\right)$$

here the constant of integration should be discarded.

Having now picked out the terms of kth degree in ε in (1.4) and (1.7), we find $t_k = t_k(x)$ and $x_k^{\cdot} = x_k^{\cdot}(x)$. Since the roots of Eqs. (1.9) and (1.10) are simple, there exists a value $\varepsilon = \varepsilon_0$ such that for all $\varepsilon < \varepsilon_0$ the roots of Eqs. (1.5) and (1.8) are simple in the kth and subsequent approximations with respect to ε [7]. Then the power series for $t_k = t_k(x)$ and $x_k^{\cdot} = x_k^{\cdot}(x)$ are uniquely determined from (1.4) and (1.7). The trajectory $x_s = x_s(x)$ constructed in this fashion depends upon three parameters: h, X_1 and X_2, which are related by Eqs. (1.5) and (1.8). As a consequence of assumption (a), from (1.6) and (1.8) we can uniquely obtain the unknown quantities as power series in ε [7] such that $h = h_0$, $X_1 = X_{10}$ and $X_2 = X_{20}$ when $\varepsilon = 0$

As a result of applying Rauscher's method the original system (1.1) is reduced to an autonomous one at each stage of the construction. The solution method is analogous to the one applied in [3] which examined the trajectories of the periodic solutions of conservative systems, close to rectilinear normal forms of oscillations. Consequently, the convergence of series (1.3) and (2.2) can be proved in some neighborhood of the origin in the same way as in [3]. Thus, to each normal form of oscillations of the unperturbed conservative system there corresponds, when conditions (a) and (b) are fulfilled, a unique resonance periodic solution of the nonautonomous system (1.1), close to the generating solution and satisfying boundary conditions (1.5), (1.6). On the trajectory of the forced steady-state mode the nonautonomous system's (1.1) behavior is similar to that of the conservative one.

We note that the generating systems cannot be linear since for the latter all roots of the periodicity Eq. (1.9) are multiple by virtue of the isochronicity property. This result emphasizes the importance of the study of the normal oscillations of nonlinear conservative systems.

3. As an example of the application of the results obtained we consider the determination of the resonance modes of the following system of equations

$$\begin{aligned} &x^{\cdot\cdot} + 4x + 2x^3 + 2.4\,(x-y) + 2\,(x-y)^3 = \varepsilon \operatorname{cn} \gamma t \\ &y^{\cdot\cdot} + 4y + 2y^3 + 2.4\,(y-x) + 2\,(y-x)^3 = 0 \end{aligned} \tag{3.1}$$

Such a type of equations arise, for example, in the problem of investigating suspension constructions whose principal operating elements are, from the point of view of calculation, flexible strings able to work only by tension.

We restrict ourselves to the examination of the "synphase" form of the steady-state oscillations in (3.1), close to the normal oscillations of an unperturbed symmetric conservative system. The zero-approximation solution has the form $y_0 = x$. In the zero approximation we choose the oscillation amplitude in such a way that the period of the natural oscillations coincides with the period of the perturbing action.

To construct the trajectory of the desired solution in the first approximation with respect to ε we use the following equation, corresponding to (2.1):

$$\frac{d^2y_1}{dx^2}[h_0 - (2x^2 + 0.5x^4)] + \frac{dy_1}{dx}[-4x - 2x^3] + \frac{x}{X_0} + 2.4y_1 = -6.4y_1 - 6x^2y_1 \qquad (3.2)$$

where X_0 is the amplitude value of x in the zero approximation, h_0 is the energy constant of the unperturbed system. We seek the solution of Eq. (3.2) in the form of series (2.2), satisfying the boundary conditions corresponding to (2.3).

For a numerical calculation we take $\gamma = 2.5$, $\varepsilon = 1$. In the generating system a solution with frequency $\omega = 2.5$ is realized for the amplitude $x(0) \equiv X = 1.225$.

In (2.2) we retain terms of no higher than the fifth degree in x. Then, by satisfying Eqs. (3.2) and boundary conditions in form (2.3), we obtain

$$y_1 \approx -0.08\,x - 0.02\,x^3 + 0.001\,x^5$$

With due regard to the zero and first approximations, we find that the form of the oscillations is

$$y \approx 0.92\,x - 0.02\,x^3 + 0.001\,x^5$$

From the periodicity conditions, corresponding to (1.8), we find that the synphase form of the resonance oscillations are realized under the following initial conditions:

$$x^{\cdot}(0) = 0,\ x(0) \equiv X = 1.38,\ y^{\cdot}(0) = 0,\ y(0) \equiv Y = 1.24$$

To estimate the accuracy of the asymptotic solution obtained, system (3.1) was investigated on an analog computer. The amplitudes of the periodic mode with the frequency of the external action $X = 1.42$, $Y = 1.34$. A comparison of the asymptotic resonance solution and the solution obtained on the analog computer shows that the calculation of the zero and the first approximations ensures an acceptable accuracy of calculation.

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